ADDENDUM

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Abstract. In the proofs of [Als21, Proposition 2.4] and [Als24, Proposition 4.8], the result from [DG, III.3.2.1] is used in the proof. This result makes a smoothness assumption which is not explicitly justified in the proofs. The aim of this note is to provide the justification. The propositions thus hold true as stated in the papers.

The main argument. Let R be a unital commutative ring.

Lemma 1. *Assume that* $\mathbf{G}$ *is an affine* R*-group scheme acting on an affine* R*-scheme* $\mathbf{X}$*, and let* $\mathbf{H}$ *be the stabilizer of a point* $x \in \mathbf{X}(R)$*. Assume further that*

1. $\mathbf{G}$ *is smooth and* $\mathbf{X}$ *is locally of finite presentation,*
2. *the fppf-quotient* $\mathbf{G}/\mathbf{H}$ *is represented by a smooth scheme, locally of finite presentation,*
3. *for every algebraically closed field* $k \in R$-**alg**, *the group* $\mathbf{G}(k)$ *acts transitively on* $\mathbf{X}(k)$*, and the point* $x_k \in \mathbf{X}(k)$ *is a regular point of the* k*-scheme* $\mathbf{X}_k$*.*

Then the monomorphism $i : \mathbf{G}/\mathbf{H} \to \mathbf{X}$ *is an isomorphism.*

The notation x_k is an instance of the following: if $\mathbf{Y}$ is an R-functor, $y \in \mathbf{Y}(R)$ and $S \in R$-**alg**, we set $y_S := \mathbf{Y}(\alpha)(y)$, where $\alpha : R \to S$ is the map that endows S with the structure of an R-ring. The monomorphism i is the one induced by the map $\mathbf{G} \to \mathbf{X}$ defined, for each $S \in R$-**alg**, by $\mathbf{G}(S) \to \mathbf{X}(S), g \mapsto g \cdot x_S$.

Proof. Since $\mathbf{G}/\mathbf{H}$ is a smooth, hence flat, R-scheme, and both $\mathbf{G}/\mathbf{H}$ and $\mathbf{X}$ are locally of finite presentation, the map i is an isomorphism if and only if this is the case over any algebraivally closed field $k \in R$-**alg**, by [EGAIV, ${}_4$17.9.5].

We may thus assume that $R = k$ is algebraically closed.

In this setting, we show $\mathbf{X}$ is smooth. It is enough to check the Jacobian criterion at each closed (hence rational) point. By assumption, $\mathbf{G}$ acts transitively on the closed points, so if y is such a point, then there is some $g \in \mathbf{G}(k)$ such that $g \cdot y$ is the regular point x. Thus g defines an automorphism of $\mathbf{X}$ defined, for each $S \in R$-**alg**, by

$$\mathbf{X}(S) \to \mathbf{X}(S), \qquad z \mapsto g_S \cdot z,$$

mapping y to a regular point. Thus y is regular. This proves that $\mathbf{X}$ is smooth, and we may apply [DG, III.3.2.1], which, given (3), implies that i is an isomorphism. □

Application to [Als21, Proposition 2.4]**.** The assumptions are satisfied with $\mathbf{G} = \mathbf{Isom}(N)$, $\mathbf{X} = \mathbf{S}_N$, $\mathbf{H} = \mathbf{Aut}(A)$ and $x = 1_A$. Over an algebraically closed field, A is split and N is given by Proposition 3.3, and $\mathbf{X}$ is the affine hypersurface given by $N(x) = 1$. In that notation, $\frac{\partial N}{\partial \alpha_1}\Big|_{1_A} = 1$, so 1_A is regular by the Jacobian criterion.

Application to [Als24, Proposition 4.8]**.** The assumptions are satisfied with $\mathbf{G} = \mathbf{Inv}(Q)$, $\mathbf{X} = \mathbf{S}_Q$, $\mathbf{H} = \mathbf{Inv}^1(Q)$ and $x = 1_B$. Over an algebraically closed field, the quartic form q is given by (2.4), and $\mathbf{X}$ is the affine hypersurface given by $q(x) = 1$. In that notation, $\frac{\partial q}{\partial r}\Big|_{1_B} = 2$, so 1_B is regular by the Jacobian criterion. (Note that 2 and 3 are assumed invertible in Proposition 4.8.)

References

[Als21] S. Alsaody, Albert algebras over rings and related torsors. Canad. J. Math. 73 (2021), 875–898.

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